

Chapter 1

Physics and Complexity: a brief spin glass perspective

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Abstract Complex macroscopic behaviour can arise in many-body systems with only very simple elements as a consequence of the combination of competition and inhomogeneity. This paper attempts to illustrate how statistical physics has driven this recognition, has contributed new insights and methodologies of wide application influencing many fields of science, and has been stimulated in return.

1.1 Introduction

Many body systems of even very simple microscopic constituents with very simple interaction rules can show novel emergence in their macroscopic behaviour. When the interactions (and any constraints) are also mutually incompatible (frustrated) and there is macroscopically relevant quenched disorder, then the emergent macroscopic behaviour can be complex (in ways to be discussed) and not simply anticipatable. Recent years have seen major advances in understanding such behaviour, in recognizing conceptual ubiquities across many apparently different systems and in forging, transferring and applying new methodologies. Statistical physics has played a major part in driving and developing the subject and in providing new methods to study and quantify it. This paper is intended to provide a brief broadbrush introduction.

A key part of these developments has been the combination of minimalist modelling, development of new concepts and techniques, and fruitful transfers of the knowledge between different systems. Here we shall concentrate on a simple paradigmatic model, demonstrate its ubiquitousness among several often very differ-

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ent systems, problems and contexts, and introduce some of the useful concepts that have arisen.

1.2 The Dean's Problem and Spin Glasses

The genesis for the explosion of interest and activity in complexity within the physics community was in an attempt to understand a group of magnetic alloys known as spin glasses¹[1]. But here we shall start with a problem that requires no physics to appreciate, the Dean's problem [2].

A College Dean is faced with the task of distributing N students between two dormitories as amicably as possible but given that some pairs of students prefer to be in the same dorm while other pairs want to be separated. If any odd number of students have an odd number of antagonistic pairwise preferences then their preferences cannot all be satisfied simultaneously. This is an example of frustration. The Dean's Problem can be modelled as a mathematical optimization problem by defining a cost function H that is to be minimized:

$$H = - \sum_{(ij)} J_{ij} \sigma_i \sigma_j; \sigma = \pm 1 \quad (1.1)$$

where the i, j label students, $\sigma = \pm 1$ indicates dorm A/B and the $\{J_{ij} = J_{ji}\}$ denote the sign and magnitude of the inter-student pair preferences ($+$ = prefer). We shall further concentrate on the situation where the $\{J_{ij}\}$ are chosen randomly and independently from a single (intensive) distribution $P(J)$ of zero mean², the random Dean's Problem. The number of combinations of possible choices grows exponentially in N (as 2^N). There is also, in general, no simple local iterative mode of solution. Hence, in general, when N becomes large the Dean's problem becomes very hard, in the language of computer science, NP-complete [3].

In fact, the cost function of the random Dean's Problem was already introduced in 1975 as a potentially soluble model for a spin glass; there it is known as the Sherrington-Kirkpatrick (SK) model [4]. In this model H is the Hamiltonian (or energy function), the i, j label spins, the σ their orientation (up/down) and the $\{J\}$ are the exchange interactions between pairs of spins.

In the latter case one was naturally interested in the effects of temperature and of phase transitions as it is varied. In the standard procedure of Gibbsian statistical mechanics, in thermal equilibrium the probability of a microstate $\{\sigma\}$ is given by

$$\mathcal{P}(\{\sigma\}) = Z^{-1} \exp[-H_{\{J_{ij}\}}(\{\sigma\})/T] \quad (1.2)$$

where Z is the partition function

¹ Spin glasses were originally observed as magnetic alloys with unusual non-periodic spin ordering. They were also later recognized as having many other fascinating glassy properties

² This restriction is not essential but represents the potentially hardest case.

$$Z = \sum_{\{\sigma\}} \exp[-H_{\{J_{ij}\}}(\{\sigma\})/T]; \quad (1.3)$$

the subscript $\{J_{ij}\}$ has been added to H to make explicit that it is for the particular instance of the (random) choice of $\{J_{ij}\}$. Again in the spirit of statistical physics one may usefully consider typical physical properties over realizations of the quenched disorder, obtainable by averaging them over those choices³.

Solving the SK model has been a great challenge and has led to new and subtle mathematical techniques and theoretical conceptualizations, backed by new computer simulational methodologies and experimentation, the detailed discussion of which is beyond the scope of this brief report. However a brief sketch will be given of some of the conceptual deductions.

Let us start pictorially. A cartoon of the situation is that of a hierarchically rugged landscape to describe the energy/cost as a function of position in the space of microscopic coordinates and such that for any local perturbations of the microscopic state that allow only downhill moves the system rapidly gets stuck and it is impossible to iterate to the true minimum or even a state close to it. Adding temperature allows also uphill moves with a probability related to $\exp[-\delta H/T]$ where δH is the energy change. But still for $T < T_g$ the system has this glassy hindrance to equilibration, a non-ergodicity that shows up, for example, in differences in response functions measured with different historical protocols.

Theoretical studies of the SK model have given this picture substance, clarification and quantification, partly by introduction of new concepts beyond those of conventional statistical physics, especially through the work of Giorgio Parisi [1, 5].

Let us assume that, at any temperature of interest, our system has possibly several essentially separate macrostates, which we label by indices $\{S\}$. A useful measure of similarity of two microstates S, S' is given by their ‘overlap’, defined as

$$q_{SS'} = N^{-1} \sum_i \langle \sigma_i \rangle_S \langle \sigma_i \rangle_{S'}. \quad (1.4)$$

where $\langle \sigma_i \rangle_S$ measures the thermal average of σ_i in macrostate S .

The distribution of overlaps is given by

$$P_{\{J_{ij}\}}(q) = \sum_{S, S'} W_S W_{S'} \delta(q - q_{SS'}), \quad (1.5)$$

where W_S is the probability of finding the system in macrostate S .

In general, the macrostates can depend on the specific choice of the $\{J_{ij}\}$ but for the SK model the average of $P_{\{J_{ij}\}}(q)$ can be calculated, as also other more complicated distributions of the $q_{SS'}$, such as the correlation of pairwise overlap distributions for 3 macrostates S, S', S'' .

For a simple (non-complex) system there is only one thermodynamically relevant macrostate and hence $\overline{P(q)}$ has a single delta function peak; at $q = 0$ for a param-

³ This is in contrast with traditional computer science which has been more concerned with worst instances.

agnet (in the absence of an external field) and at $q = m^2$ for a ferromagnet, where m is the magnetization per spin. In contrast, in a complex system $\overline{P(q)}$ has structure indicating many relevant macrostates⁴. This is the case for the SK model beneath a critical temperature and for sufficient frustration, as measured by the ratio of the standard deviation of $P(J)$ compared with its mean. Furthermore other measures of the q -distribution indicate a hierarchical structure, ultrametricity and a phylogenetic-tree structure for relating overlaps of macrostates, chaotic evolution with variations of global parameters, and also non-self-averaging of appropriate measures.

These observations and others give substance to and quantify the rugged landscape picture with macrostate barriers impenetrable on timescales becoming infinite with N . For finite-ranged spin glasses this picture must be relaxed to have only finite barriers, but still with a non-trivial phase transition to a glassy state.

The macroscopic dynamics in the spin glass phase also shows novel and interesting glassy behaviour⁵, never equilibrating and having significant deviations from the usual fluctuation-dissipation relationship⁶.

A brief introduction to the methodologies to arrive at these conclusions is deferred to a later section.

1.3 Transfers and Extensions

The knowledge gained from such spin glass studies has been applied to increasing understanding of several other physically different systems and problems, via mathematical and conceptual transfers and extensions. Conversely these other systems have presented interesting new challenges for statistical physics. In this section we shall illustrate this briefly via discussion of some of these transfers and stimulating extensions.

In static/thermodynamic extensions there exist several different analogues of the quenched and annealed microscopic variables, $\{J\}$ and $\{\sigma\}$ above, and of the intensive controls, such as T . Naturally, in dynamics of systems with quenched disorder the annealed variables (such as the $\{\sigma\}$ above) become dynamical, but also one can consider cases in which the previously quenched parameters are also dynamical but with slower fundamental microscopic timescales⁷.

⁴ The overline indicates an average over the quenched disorder.

⁵ There are several possible microscopic dynamics that leads to the same equilibrium/Gibbsian measure, but all such employing local dynamics lead to glassiness.

⁶ Instead one finds a modified fluctuation-dissipation relation with the temperature normalized by the instantaneous auto-correlation.

⁷ Sometimes one speaks of fast and slow microscopic variables but it should be emphasised that these refer to the underlying microscopic time-scales. Glassiness leads to much slower macroscopic timescales.

1.3.1 Optimization and satisfiability

Already in section (1.2), one example of eqn.(1.1) as an optimization problem was given (the Dean's Problem). Another classic hard computer science optimization problem is that of equipartitioning a random graph so as to minimise the cross-links. In this case the cost function to minimise can be again be written as in eqn. (1.1), now with the $\{i\}$ labelling vertices of the graph, the $\{J_{ij}\}$ equal to 1 on edges/graph-links between vertices and zero where there is no link between i and j , the $\{\sigma_i = \pm\}$ indicating whether vertices i are in the first or second partition and with the frustrating constraint $\sum_i \sigma_i = 0$ imposing equipartitioning. Without the global constraint this is a random ferromagnet, but with it the system is in the same complexity class as a spin glass.

Another classic hard optimization problem that extends eqn. (1.1) in an apparently simple way but in fact leads to new consequence is that of random K -satisfiability (K -SAT) [11]. Here the object is to investigate the simultaneous satisfiability of many, M , randomly chosen clauses, each made up of K possible microscopic conditions involving a large number, N , of binary variables. Labelling the variables $\{\sigma_i\} = \{\pm 1\}$ and writing x_i to indicate $\sigma_i = 1$ and \bar{x}_i to indicate $\sigma_i = -1$, a K -clause has the form

$$(y_{i_1} \text{ or } y_{i_2} \text{ or } \dots y_{i_K}); i = 1, \dots M \quad (1.6)$$

where the y_{i_j} are x_{i_j} or \bar{x}_{i_j} . In Random K -Sat the $\{i_j\}$ are chosen randomly from the N possibilities and the choice of $y_{i_j} = x_{i_j}$ or \bar{x}_{i_j} is also random, in both cases then quenched. In this case one finds, for the thermodynamically relevant typical system, that there are two transitions as the ratio $\alpha = M/N$ is increased in the limit $N \rightarrow \infty$; for $\alpha > \alpha_{c1}$ it is not possible to satisfy all the clauses simultaneously (UNSAT), for $\alpha < \alpha_{c1}$ the problem is satisfiable in principle (SAT), but for $\alpha_{c2} < \alpha < \alpha_{c1}$ it is very difficult to satisfy (in the sense that all simple local variational algorithms stick) and this region is known as HARD-SAT. These distinctions are attributable to regions of fundamentally different fractionation of the space of satisfiability, different levels of complexity.

1.3.2 K -spin glass

In fact, again there was a stimulating precursor of this K -SAT discovery in a “*what-if*” extension of the SK model [6, 7] in which the 2-spin interactions of eqn. (1.1) are replaced by K -spin interactions:

$$H_K = - \sum_{(i_1, i_2, \dots, i_K)} J_{i_1, i_2, \dots, i_K} \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_K} \quad (1.7)$$

in which the J_{i_1, i_2, \dots, i_K} are again chosen randomly and independently from an intensive distribution of zero mean. In this case, two different phase transitions are observed as a function of temperature, a lower thermodynamic transition and a dynamical transition that is at a slightly higher temperature, both to complex spin glass phases. The thermodynamic transition represents what is achievable in principle in a situation in which all microstates can be accessed; the dynamical transition represents the situation where the system gets stuck and cannot explore all the possibilities, analogues of HARDSAT-UNSAT and SAT-HARDSAT.

The K -spin glass is also complex with a non-trivial overlap distribution function $P(q)$ but now the state first reached as the transitions are crossed has a different structure from that found for the 2-spin case. Now

$$\overline{P(q)} = (1 - x)\delta(q - q_{\min}) + x\delta(q - q_{\max}); \quad (1.8)$$

in contrast with the SK case where there is continuous weight below the maximum q_{\max} . The two delta functions demonstrate that there is still the complexity of many equivalent but different macrostates, but now with equal mutual orthogonalities (as compared with the 2-spin SK case where there is a continuous range of macrostates). This situation turns out to be quite common in many extensions beyond SK.

1.3.3 Statics, dynamics and temperature

At this point it is perhaps useful to say a few more words about the differences between statics/thermodynamics and dynamics in statistical physics, and about types of micro-dynamics and analogues of temperature.

In a physical system one often wishes to study thermodynamic equilibrium, assuming all microstates are attainable if one waits long enough. In optimization problems one typically has two types of problem; the first determining what is attainable in principle, the second considering how to attain it. The former is the analogue of thermodynamic equilibrium, the latter of dynamics.

In a physical system the true microscopic dynamics is given by nature. However, in optimization studies the investigator has the opportunity to determine the micro-dynamics through the computer algorithms he or she chooses to employ.

Temperature enters the statistical mechanics of a physical problem in the standard Boltzmann-Gibbs ensemble fashion, or as a measure of the stochastic noise in the dynamics. We have already noted that it can also enter an optimization problem in a very similar fashion if there is inbuilt uncertainty in the quantity to be optimized. But stochastic noise can also usefully be introduced into the artificial computer algorithmic dynamics used to try to find that optimum. This is the basis of the optimization technique of simulated annealing where noise of variance T_A is deliberately introduced to enable the probabilistic scaling of barriers, and then gradually reduced to zero [8].

1.3.4 Neural networks

The brain is made up of a very large number of neurons, firing at different rates and extents, interconnected by an even much larger number of synapses, both excitatory and inhibitory. In a simple model due to Hopfield [9] one can consider a cartoon describable again by a control function of the form of eqn. (1.1). In this model the neurons $\{i\}$ are idealized by binary McCulloch-Pitts variables $\{\sigma_i = \pm 1\}$, the synapses by $\{J_{ij}\}$, positive for excitatory and negative for inhibitory, with stochastic neural microdynamics of effective temperature T_{neural} emulating the width of the sigmoidal response of a neuron's output to the combined input from all its afferent synapses, weighted by the corresponding activity of the afferent neurons.

The synapses are distributed over both signs, yielding frustration and apparently random at first sight. However actually they are coded to enable attractor basins related to memorized patterns of the neural microstates $\{\xi_i^\mu\}$; $\mu = 1, \dots, p = \alpha N$. The similarity of a neural micro-state to a pattern μ is given by an overlap

$$m^\mu = N^{-1} \sum_i \langle \sigma_i \rangle_S \xi_i^\mu. \quad (1.9)$$

Retrieval of memory μ is the attractor process in which a system started with a small m^μ iterates towards a large value of m^μ .

In Hopfield's original model he took the $\{J_{ij}\}$ to be given by the Hebb-inspired form

$$J_{ij} = p^{-1} \sum_\mu \xi_i^\mu \xi_j^\mu \quad (1.10)$$

with randomly quenched $\{\xi_i\}$ ⁸. For α less than a T_{neural} -dependent critical value $\alpha_c(T_{\text{neural}})$ patterns can be retrieved. Beyond it only quasi-random spin glass minima unrelated to the memorised patterns remain (and still only for T_{neural} not too large). However, other $\{J_{ij}\}$ permit a slightly larger capacity (as also can occur for correlated patterns).

Again the landscape cartoon is illustratively useful. It can be envisaged as one for $H_{\{J_{ij}\}}$ as a function of the neural microstates (of all the neurons), with the dynamics one of motion in that landscape, searching for minima using local deviation attempts. The memory basins are large minima. Clearly one would like to have many different retrievable memories. Hence frustration is necessary. But equally, too much frustration would lead to a spin-glass-like state with minima unrelated to learned memories.

This cartoon also leads immediately to the recognition that learning involves modifying the landscape so as to place the attractor minima around the states to be retrieved. This extension can be modelled minimally via a system of coupled dynamics of neurons whose state dynamics is fast (attempting retrieval or generalization) and synapses that also vary dynamically but on a much slower timescale and in response to external perturbations (yielding learning) [10].

⁸ *i.e.* uncorrelated patterns

1.3.5 Minority game

More examples of many-body systems with complex macrobehaviour are to be found in social systems, in which the microscopic units are people (or groups of people or institutions), sometimes co-operating, often competing. Here explicit discussion will be restricted to one simple model, the Minority Game [12], devised to emulate some features of a stockmarket. N ‘agents’ play a game in which at each time-step each agent makes one of two choices with the objective to make the choice which is in the minority⁹. They have no direct knowledge of one another but (in the original version) make their choices based on the commonly-available knowledge of the historical actual minority choices, using their own individual strategies and experience to make their own decisions. In the spirit of minimalism we consider all agents (i) to have the same ‘memories’, of the minority choices for the last m time-steps, (ii) to each have two strategies given by randomly chosen and quenched Boolean operators that, acting on the m -string of binary entries representing the minority choices for the last m steps, output a binary instruction on the choice to make, (iii) using a personal ‘point-score’ to keep tally of how their strategies would have performed if used, increasing the score each time they would have chosen the actual minority, and (iv) using their strategy with the larger point-score. Frustration is represented in the minority requirement, while quenched disorder arises in the random choice of individual strategies.

Simulation studies of the ‘volatility’, the standard deviation of the actual minority choice, shows (i) a deviation from individually random choices, indicating correlation through the common information, (ii) a cusp-minimum at a critical value α_c of the ratio of the information dimension to the number of agents $\alpha = D/N = 2^m/N$, suggesting a phase transition at α_c , (iii) ergodicity for $\alpha > \alpha_c$ but non-ergodic dependence on the point-score initialization for $\alpha < \alpha_c$, indicating that the transition represents the onset of complexity. This is reminiscent of the cusp and the ergodic-nonergodic transition observed in the susceptibilities of spin glass systems as the temperature is reduced through the spin glass transition.

Furthermore, this behaviour is essentially unaltered if the ‘true’ history is replaced by a fictitious ‘random’ history at each step, with all agents being given the same false history, indicating that it principally represents a carrier for an effective interaction between the agents. Indeed, generalising to a D -dimensional random history information space, considering this as a vector-space and the strategies as quenched D -vectors of components $\{R_i^{s,\mu}\}$; $s = 1, 2$, $\mu = 1, \dots, D$ in that space, and averaging over the stochastically random ‘information’, one is led to an effective control function analogous to those of eqn. (1.1) and eqn. (1.10) with p replaced by α , now $\{\xi_i = (R_i^1 - R_i^2)/2\}$, an extra multiplicative minus sign on the right hand side of eqn. (1.10), and also a random-field term dependent upon the $\{\xi_i\}$ and $\{\omega_i = (R_i^1 + R_i^2)/2\}$. As noted, there is an ergodic-nonergodic transition at a cr-

⁹ The philosophy is that one gets the best price by selling when most want to buy or buying when most want to sell.

titical α , but now the picture is one of the $\{\xi_i\}$ as repellers rather than the attractors of the Hopfield model¹⁰.

The typical behaviour of this system, as for the spin glasses, can be studied using a dynamical generating functional method [13], averaged over the choice of quenched strategies, in a manner outlined below. The averaged many-body system can then be mapped into an effective *representative agent ensemble* with *memory* and *coloured noise*, with both the noise correlations and the memory kernel determined self-consistently over the ensemble. Note that this is in contrast to (and corrects) the common assumption of a single deterministic representative agent. The phase transition from ergodic to non-ergodic is manifest by a singularity in the two-time point-sign correlation function

$$C(t, t') = N^{-1} \sum_i \overline{\text{sgn}(p_i(t) \text{sgn}(p_i(t'))} = \langle \text{sgn}(p(t) \text{sgn}(p(t')) \rangle_{ens} \quad (1.11)$$

where the first equality refers to the many-body problem and the second its equivalence in the effective agent ensemble.

1.4 Methodologies

For systems in equilibrium, physical observables are given by $\ln Z$ evaluated for the specific instance of any quenched parameters, or strictly the generalized generating function $\ln Z(\{\lambda\})$ where the $\{\lambda\}$ are generating fields to be taken to zero after an appropriate operation (such as $\partial/\partial\lambda$) is performed. Hence the average over quenched disorder is given by $\overline{\ln Z}$. One would like to perform the average over quenched disorder explicitly to yield an effective system. However, since Z is a sum over exponentials of a function of the variables, $\ln Z$ is difficult to average directly so instead one uses the relation

$$\ln Z = \lim_{n \rightarrow 0} n^{-1} (Z^n - 1) \quad (1.12)$$

and interprets the Z^n as corresponding to a system whose variables have extra ‘replica’ labels, $\alpha = 1, \dots, n$, for which one can then average the partition function, an easier operation, at the price of needing to take the eventual limit $n \rightarrow 0$. The relevant ‘order parameters’ are then correlations between replicas

$$q^{\alpha\beta} = N^{-1} \sum_i \langle \sigma_i^\alpha \sigma_i^\beta \rangle_T \quad (1.13)$$

¹⁰ One can make the model even more minimal by allowing each agent only one strategy $\{\xi_i\}$ which (s)he either follows if its point-score is positive or acts oppositely to if the point-score is negative. This removes the random-field term and also the cusp in the *tabula rasa* volatility, but retains the ergodic-nonergodic transition[14].

where $\langle \dots \rangle_T$ refers to a thermal average in the effective post-averaging system. This order parameter is non-zero in the presence of frozen order, but more interestingly (and subtly) also exhibits the further remarkable feature of spontaneous replica symmetry breaking, indicating complexity. After further subtleties beyond the scope of this short introduction, there emerges an order function $q(x); x \in [0,1]$ from which the average overlap function is obtained by

$$\overline{P(q)} = dx/dq \quad (1.14)$$

For dynamics the analogue of the partition function Z is a generating functional, which may be written symbolically as

$$Z_{dyn} = \int \prod_{\text{all variables, all times}} \delta(\text{microscopic eqns. of motion}) \exp(\{\lambda \phi\}) \quad (1.15)$$

where the ϕ symbolize the microscopic variables and a Jacobian is implicit. Averaging over the quenched disorder now induces interaction between epochs and integrating out the microscopic variables results in the effective single agent ensemble formulation, as well as emergent correlation and response functions as the dynamic order parameter analogues of the static inter-replica overlaps, exhibiting non-analyticity at a phase transition to non-ergodicity.

1.5 Conclusion

A brief illustration has been presented of how complex co-operative behaviour arises in many body systems due to the combination of frustration and disorder in the microscopics of even very simply formulated problems with very few parameters. Such systems are not only examples of Anderson's famous quotation "*More is different*" but also demonstrate that *frustration and disorder in microscopics can lead to complexity in macroscopics*; i.e. many and complexly related *different*s. Furthermore, this complexity arises in systems with very simple few-valued microscopic parameters; *complexity is **not** the same as complication* and does not require it.

There has also been demonstrated valuable transfers between systems that appear very different at first sight, through the media of mathematical modelling, conceptualization and investigatory methodologies, a situation reminiscent of the successful use of the Rosetta stone in learning an unknown language script by comparison with another that carries the same message in a different format.

The perspective taken has been of statistical physics, but it must be emphasised that the stimulation has been both from and to physics, since many of these complex systems have interesting features in their microscopic underpinning that are richer than those in the physics of conventional dictionary definition and provide new challenges to the physicist.

Also of note is how a *blue skies* attempt to understand some obscure magnetic alloys through soluble but, for the experimental alloys, unphysical modelling has

led to an explosion of appreciation of new concepts, understanding and application of ideas and methodologies throughout an extremely wide range of the sciences.

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